The Spectrum of $\Box_b^t$ on the 3-sphere

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Proof Writing

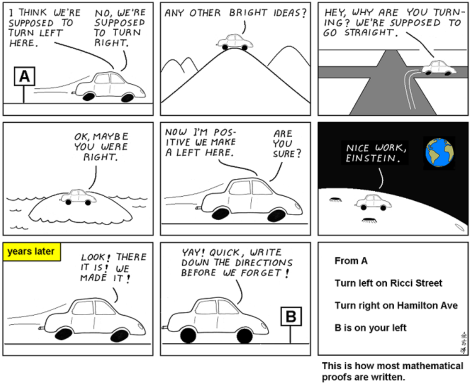

Figure: Obvious, by Abstruse Goose. https://abstrusegoose.com/230

Problem and Results

Question

Is there a sequence of eigenvalues in the spectrum of $\Box_b^t$ that converges to 0?

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Our Result

Yes! The smallest eigenvalue of $\Box_b^t$ on $\mathcal{H}_{2k-1}(\mathbb{S}^3)$, $\lambda_{min,2k-1}$, is bounded above by

$$\lambda_{min,2k-1} \leq \frac{1+|t|^2}{(1-|t|^2)^2}(2k-1)\sqrt{k}|t|^{2k}$$

which goes to 0 as $k \to \infty$.

Complex Polynomials

Before we can talk about $\Box_b^t$, we have to talk about complex polynomial spaces. A complex polynomial in $\mathbb{C}^2$ is a polynomial with coefficients in $\mathbb{C}$ in unknowns $z_1, z_2, \overline{z_1}, \overline{z_2}$. Some examples are $2z_1 + z_2\overline{z_2}, 3z_1^2 z_2 - z_2^3$, and $6\overline{z_1}^2\overline{z_2} + 3z_1^2$.

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When talking about the degree of complex polynomials, we use the bidegree p, q, where p is the degree of the non-conjugated terms and q is the degree of the conjugated terms. As an example, $z_1 z_2^2 \overline{z_2}$ has bidegree $3, 1$. When we say the degree, it is just the sum $p + q$ of the bidegree.

Harmonic and Homogeneous Polynomials

A homogeneous polynomial is a polynomial where the bidegree of every term is the same. So $z_1^2 - 3z_1z_2$ is homogeneous (bidegree 2,0), but $2z_2 - 3\overline{z_1}^2$ is not.

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A harmonic polynomial is a polynomial whose Laplacian is 0. In $\mathbb{C}^2$, this is equivalent to saying that

$$\Delta p = 4\left(\frac{\partial^2 p}{\partial z_1 \partial \overline{z_1}} + \frac{\partial^2 p}{\partial z_2 \partial \overline{z_2}}\right) = 0$$

One example is $4z_1z_2\overline{z_1} - 2z_2^2\overline{z_2}$.

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Using these properties, we can define some spaces:

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We can also talk about these spaces over $\mathbb{S}^3$, which is the restriction of the polynomials in $\mathbb{C}^2$ to $\mathbb{S}^3$.

Basis for $\mathcal{H}_k(\mathbb{S}^3)$

We can compute a basis for $\mathcal{H}_k(\mathbb{S}^3)$ with the following theorems:

Theorem 1

The set

$$\left\{ \overline{D}^{\alpha} D^{\beta} |z|^{-2} \;\middle|\; |\alpha| = p, |\beta| = q, \alpha_1 = 0 \text{ or } \beta_2 = 0 \right\}$$

is an orthogonal basis for $\mathcal{H}_{p,q}(\mathbb{S}^3)$.

Theorem 2

$$\mathcal{H}_k(\mathbb{S}^3) = \bigoplus_{p+q=k} \mathcal{H}_{p,q}(\mathbb{S}^3)$$

Defining $\Box_b^t$

With these definitions, we now define

$$\mathcal{L} = \overline{z_2}\frac{\partial}{\partial z_1} - \overline{z_1}\frac{\partial}{\partial z_2} \qquad \overline{\mathcal{L}} = z_2\frac{\partial}{\partial \overline{z_1}} - z_1\frac{\partial}{\partial \overline{z_2}}$$

These operate on $L^2(\mathbb{S}^3)$, the space of square-integrable functions on the 3-sphere in $\mathbb{C}^2$.

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These operate on $L^2(\mathbb{S}^3)$, the space of square-integrable functions on the 3-sphere in $\mathbb{C}^2$. Then our operator $\Box_b^t$ is defined as

$$\Box_b^t = -(\mathcal{L} + \bar{t}\overline{\mathcal{L}})\left(\frac{1+|t|^2}{(1-|t|^2)^2}\right)(\overline{\mathcal{L}} + t\mathcal{L})$$

where t is a complex number with $|t| < 1$. We note that $\Box_b^t$ is linear and self-adjoint with this definition, so all its eigenvalues are real.

Why Polynomial Spaces?

We have a theorem that states

Theorem 3

$$L^2(\mathbb{S}^3) = \bigoplus_{k=1}^{\infty} \mathcal{H}_k(\mathbb{S}^3)$$

so instead of studying our operator on $L^2(\mathbb{S}^3)$, we can study it on the finite-dimensional slices $\mathcal{H}_k(\mathbb{S}^3)$, which we have an orthogonal basis for.
Since $\Box_b^t$ ends up being invariant on these $\mathcal{H}_k(\mathbb{S}^3)$, we can compute the matrix representation of $\Box_b^t$ on these spaces, and use that to find its eigenvalues.

Our Goals

As mentioned at the beginning, we need to find a sequence of eigenvalues in the spectrum of $\Box_b^t$ that goes to 0. With all of this, how do we actually get there?

1. Compute the bases of the spaces $\mathcal{H}_{p,q}(\mathbb{S}^3)$ and $\mathcal{H}_k(\mathbb{S}^3)$.

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1. Compute the bases of the spaces $\mathcal{H}_{p,q}(\mathbb{S}^3)$ and $\mathcal{H}_k(\mathbb{S}^3)$.
2. Compute the matrix representation of $\Box_b^t$ over $\mathcal{H}_k(\mathbb{S}^3)$.

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1. Compute the bases of the spaces $\mathcal{H}_{p,q}(\mathbb{S}^3)$ and $\mathcal{H}_k(\mathbb{S}^3)$.
2. Compute the matrix representation of $\Box_b^t$ over $\mathcal{H}_k(\mathbb{S}^3)$.
3. Compute the eigenvalues of this matrix, and find a sequence of eigenvalues that goes to 0.
4. Actually prove that this sequence exists, which is most of the work.

Doing the Computation

We used Mathematica for nearly all of the computation we needed. After spending a couple weeks learning it, we were able to produce the bases for $\mathcal{H}_k(\mathbb{S}^3)$ and find the matrix representation and eigenvalues of $\Box_b^t$ on these spaces. So what do they look like?

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We used Mathematica for nearly all of the computation we needed. After spending a couple weeks learning it, we were able to produce the bases for $\mathcal{H}_k(\mathbb{S}^3)$ and find the matrix representation and eigenvalues of $\Box_b^t$ on these spaces. So what do they look like?

```
In[40]:= H2

Out[40]= {2 Conjugate[z2]^2, 2 Conjugate[z1] Conjugate[z2], 2 Conjugate[z1]^2,
  -Conjugate[z1] z1 + Conjugate[z2] z2, 2 Conjugate[z1] z2, 2 Conjugate[z2] z1, 2 z2^2, 2 z1 z2, 2 z1^2}

In[41]:= H3

Out[41]= {-6 Conjugate[z2]^3, -6 Conjugate[z1] Conjugate[z2]^2, -6 Conjugate[z1]^2 Conjugate[z2],
  -6 Conjugate[z1]^3, 4 Conjugate[z1] Conjugate[z2] z1 - 2 Conjugate[z2]^2 z2,
  2 Conjugate[z1]^2 z1 - 4 Conjugate[z1] Conjugate[z2] z2, -6 Conjugate[z1]^2 z2,
  -6 Conjugate[z2]^2 z1, 4 Conjugate[z1] z1 z2 - 2 Conjugate[z2] z2^2, -6 Conjugate[z1] z2^2,
  2 Conjugate[z1] z1^2 - 4 Conjugate[z2] z1 z2, -6 Conjugate[z2] z1^2, -6 z2^3, -6 z1 z2^2, -6 z1^2 z2, -6 z1^3}
```

Doing the Computation

In[1512]:= **BuildMatrix[B, A₂]**

Out[1512]//MatrixForm=

$$\begin{pmatrix}
-\frac{2(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\,\mathrm{Conjugate}[t](1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2} \\
0 & -\frac{2(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & -\frac{2\,\mathrm{Conjugate}[t](1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2} & 0 \\
0 & 0 & -\frac{2(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2} & 0 & 0 & 0 & \frac{2\,\mathrm{Conjugate}[t](1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2} & 0 & 0 \\
0 & 0 & 0 & -\frac{2(1+t\,\mathrm{Conjugate}[t])(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{2(1+t\,\mathrm{Conjugate}[t])(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{2(1+t\,\mathrm{Conjugate}[t])(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2} & 0 & 0 & 0 \\
0 & 0 & \frac{2t(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2} & 0 & 0 & 0 & -\frac{2t\,\mathrm{Conjugate}[t](1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2} & 0 & 0 \\
0 & -\frac{2t(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & -\frac{2t\,\mathrm{Conjugate}[t](1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2} & 0 \\
\frac{2t(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2t\,\mathrm{Conjugate}[t](1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2}
\end{pmatrix}$$

In[1513]:= **Eigenvalues[%1512]**

Out[1513]=

$$\left\{0, 0, 0, \frac{2(-1-t\,\mathrm{Conjugate}[t])(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2}, \frac{2(-1-t\,\mathrm{Conjugate}[t])(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2}, \frac{2(-1-t\,\mathrm{Conjugate}[t])(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2}, \frac{2(-1-t\,\mathrm{Conjugate}[t])(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2}, \frac{2(-1-t\,\mathrm{Conjugate}[t])(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2}, \frac{2(-1-t\,\mathrm{Conjugate}[t])(1+\mathrm{Norm}[t]^2)}{-1+\mathrm{Norm}[t]^2}\right\}$$

In[1514]:= **T[Eigenvalues[%1512], .05]**

Out[1514]= {0, 0, 0, 2.01505, 2.01505, 2.01505, 2.01505, 2.01505, 2.01505}

Doing the Computation

In[1515]:= **BuildMatrix[B, A_3]**

Out[1515]=

$$\left(\begin{array}{cccccccccccccccc}
-\frac{3(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6\,\text{Conjugate}[t](1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 \\
0 & -\frac{3(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{6\,\text{Conjugate}[t](1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{3(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & \frac{6\,\text{Conjugate}[t](1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{3(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & \frac{6\,\text{Conjugate}[t](1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{(4+3t\,\text{Conjugate}[t])(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\,\text{Conjugate}[t](1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{(4+3t\,\text{Conjugate}[t])(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2\,\text{Conjugate}[t](1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{(4+3t\,\text{Conjugate}[t])(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & \frac{2\,\text{Conjugate}[t](1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{(4+3t\,\text{Conjugate}[t])(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\,\text{Conjugate}[t](1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} \\
0 & 0 & \frac{2t(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & -\frac{(3+4t\,\text{Conjugate}[t])(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2t(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & -\frac{(3+4t\,\text{Conjugate}[t])(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{2t(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{(3+4t\,\text{Conjugate}[t])(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 \\
\frac{2t(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{(3+4t\,\text{Conjugate}[t])(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{6t(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & -\frac{3t\,\text{Conjugate}[t](1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{6t(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3t\,\text{Conjugate}[t](1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{6t(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3t\,\text{Conjugate}[t](1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6t(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3t\,\text{Conjugate}[t](1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2}
\end{array}\right)$$

In[1516]:= **Eigenvalues[%1515]**

Out[1516]=

$$\Big\{\frac{\left(-2-3t\,\text{Conjugate}[t]-2\sqrt{1+3t\,\text{Conjugate}[t]}\right)(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2}, \frac{\left(-2-3t\,\text{Conjugate}[t]-2\sqrt{1+3t\,\text{Conjugate}[t]}\right)(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2}, \frac{\left(-2-3t\,\text{Conjugate}[t]-2\sqrt{1+3t\,\text{Conjugate}[t]}\right)(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2}, \frac{\left(-2-3t\,\text{Conjugate}[t]-2\sqrt{1+3t\,\text{Conjugate}[t]}\right)(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2},$$

$$\frac{\left(-2-3t\,\text{Conjugate}[t]+2\sqrt{1+3t\,\text{Conjugate}[t]}\right)(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2}, \frac{\left(-2-3t\,\text{Conjugate}[t]+2\sqrt{1+3t\,\text{Conjugate}[t]}\right)(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2}, \frac{\left(-2-3t\,\text{Conjugate}[t]+2\sqrt{1+3t\,\text{Conjugate}[t]}\right)(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2}, \frac{\left(-2-3t\,\text{Conjugate}[t]+2\sqrt{1+3t\,\text{Conjugate}[t]}\right)(1+\text{Norm}[t]^2)}{-1+\text{Norm}[t]^2},$$

$$\frac{\left(-6-4t\,\text{Conjugate}[t]-\sqrt{-36+(6+4t\,\text{Conjugate}[t])^2}\right)(1+\text{Norm}[t]^2)}{2(-1+\text{Norm}[t]^2)}, \frac{\left(-6-4t\,\text{Conjugate}[t]-\sqrt{-36+(6+4t\,\text{Conjugate}[t])^2}\right)(1+\text{Norm}[t]^2)}{2(-1+\text{Norm}[t]^2)}, \frac{\left(-6-4t\,\text{Conjugate}[t]-\sqrt{-36+(6+4t\,\text{Conjugate}[t])^2}\right)(1+\text{Norm}[t]^2)}{2(-1+\text{Norm}[t]^2)}, \frac{\left(-6-4t\,\text{Conjugate}[t]-\sqrt{-36+(6+4t\,\text{Conjugate}[t])^2}\right)(1+\text{Norm}[t]^2)}{2(-1+\text{Norm}[t]^2)},$$

$$\frac{\left(-6-4t\,\text{Conjugate}[t]+\sqrt{-36+(6+4t\,\text{Conjugate}[t])^2}\right)(1+\text{Norm}[t]^2)}{2(-1+\text{Norm}[t]^2)}, \frac{\left(-6-4t\,\text{Conjugate}[t]+\sqrt{-36+(6+4t\,\text{Conjugate}[t])^2}\right)(1+\text{Norm}[t]^2)}{2(-1+\text{Norm}[t]^2)}, \frac{\left(-6-4t\,\text{Conjugate}[t]+\sqrt{-36+(6+4t\,\text{Conjugate}[t])^2}\right)(1+\text{Norm}[t]^2)}{2(-1+\text{Norm}[t]^2)}, \frac{\left(-6-4t\,\text{Conjugate}[t]+\sqrt{-36+(6+4t\,\text{Conjugate}[t])^2}\right)(1+\text{Norm}[t]^2)}{2(-1+\text{Norm}[t]^2)}\Big\}$$

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There are a couple things to note here:

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There are a couple things to note here:

1. This matrix is a mess.
2. The matrix is more structured than the definition makes it out to be. We also have a lot of repeated eigenvalues. Why?
3. In our case, it took far too long to get these matrices for anything higher than $\mathcal{H}_4(\mathbb{S}^3)$: sometimes upward of 10 minutes. For each entry, we had to compute an L^2 inner product, which is

$$\langle p, q \rangle = \int_{\mathbb{S}^3} p\overline{q}\, d\sigma$$

which is very taxing on the computer.

Solving the First Problems

To make the matrix less messy, we took the constant $\frac{1+|t|^2}{(1-|t|^2)^2}$ out of it. This left us with matrices that look like this.

```
In[42]:= MatrixForm[BuildMatrix[BoxBTU, H2]]

Out[42]//MatrixForm=
( 2    0    0    0                         0                         0                         0                   0                   -2 Conjugate[t] )
( 0    2    0    0                         0                         0                         0                   2 Conjugate[t]      0               )
( 0    0    2    0                         0                         0                         -2 Conjugate[t]     0                   0               )
( 0    0    0    2 (1 + t Conjugate[t])    0                         0                         0                   0                   0               )
( 0    0    0    0                         2 (1 + t Conjugate[t])    0                         0                   0                   0               )
( 0    0    0    0                         0                         2 (1 + t Conjugate[t])    0                   0                   0               )
( 0    0    -2 t 0                         0                         0                         2 t Conjugate[t]    0                   0               )
( 0    2 t  0    0                         0                         0                         0                   2 t Conjugate[t]    0               )
( -2 t 0    0    0                         0                         0                         0                   0                   2 t Conjugate[t])
```

While this is much clearer, it didn't make the computation any faster, so we had to do something different for that.

Solving the First Problems

We noticed that in the original definition of $\Box_b^t$, that because $\mathcal{L}$ and $\overline{\mathcal{L}}$ are linear, we can distribute and simplify it as

$$\Box_b^t = -\frac{1+|t|^2}{(1-|t|^2)^2}(\mathcal{L}\overline{\mathcal{L}} + |t|^2\overline{\mathcal{L}}\mathcal{L} + t\mathcal{L}^2 + \overline{t}\overline{\mathcal{L}}^2)$$

so we can compute the matrix representations of $\mathcal{L}\overline{\mathcal{L}}, \overline{\mathcal{L}}\mathcal{L}, \mathcal{L}^2, \overline{\mathcal{L}}^2$ individually and recombine them using this formula to get the matrix.

Solving the First Problems

```
In[45]:= MatrixForm[SingleEntryMatrix[LOperator @* LBarOperator, H2]]
```

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[47]:= MatrixForm[SingleEntryMatrix[LOperator @* LOperator, H2]]
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[46]:= MatrixForm[SingleEntryMatrix[LBarOperator @* LOperator, H2]]
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

```
In[48]:= MatrixForm[SingleEntryMatrix[LBarOperator @* LBarOperator, H2]]
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using the fact that each of these matrices has only a single entry in each row/column, we were able to speed up the computation significantly by avoiding the computation of the inner product.

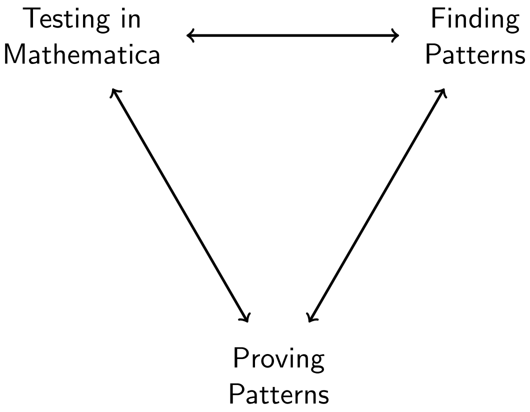
Solving the First Problems

In[49]:= `MatrixForm[BoxBTUMatrix[H2]]`

2	0	0	0	0	0	0	0	-2 t
0	2	0	0	0	0	0	2 t	0
0	0	2	0	0	0	-2 t	0	0
0	0	0	2 + 2 t Conjugate[t]	0	0	0	0	0
0	0	0	0	2 + 2 t Conjugate[t]	0	0	0	0
0	0	0	0	0	2 + 2 t Conjugate[t]	0	0	0
0	0	-2 Conjugate[t]	0	0	0	2 t Conjugate[t]	0	0
0	2 Conjugate[t]	0	0	0	0	0	2 t Conjugate[t]	0
-2 Conjugate[t]	0	0	0	0	0	0	0	2 t Conjugate[t]

It was around this time we noticed that the matrices we were getting were the transpose of the real matrix, so we corrected it in this new function.
While this matrix is nicer, there is still not enough information to prove what the eigenvalues are. Can we find something else that will help us simplify it further?

How Do I Prove Things?



Pattern Hunting

We noticed in the matrix of $\Box_b^t$ on $\mathcal{H}_3(\mathbb{S}^3)$ that the entries seemed to line up in an interesting way:

```
In[43]:= MatrixForm[BuildMatrix[BoxBTU, H3]]
Out[43]//MatrixForm=
```

3	0	0	0	0	0	0	0	0	0	0	-6 Conjugate[t]	0	0	0	0
0	3	0	0	0	0	0	0	0	0	6 Conjugate[t]	0	0	0	0	0
0	0	3	0	0	0	0	0	-6 Conjugate[t]	0	0	0	0	0	0	0
0	0	0	3	0	0	0	0	0	-6 Conjugate[t]	0	0	0	0	0	0
0	0	0	0	4 + 3 t Conjugate[t]	0	0	0	0	0	0	0	0	0	-2 Conjugate[t]	0
0	0	0	0	0	4 + 3 t Conjugate[t]	0	0	0	0	0	0	0	2 Conjugate[t]	0	0
0	0	0	0	0	0	4 + 3 t Conjugate[t]	0	0	0	0	0	-2 Conjugate[t]	0	0	0
0	0	0	0	0	0	0	4 + 3 t Conjugate[t]	0	0	0	0	0	0	0	-2 Conjugate[t]
0	0	-2 t	0	0	0	0	0	3 + 4 t Conjugate[t]	0	0	0	0	0	0	0
0	0	0	-2 t	0	0	0	0	0	3 + 4 t Conjugate[t]	0	0	0	0	0	0
0	2 t	0	0	0	0	0	0	0	0	3 + 4 t Conjugate[t]	0	0	0	0	0
-2 t	0	0	0	0	0	0	0	0	0	0	3 + 4 t Conjugate[t]	0	0	0	0
0	0	0	0	0	0	-6 t	0	0	0	0	0	3 t Conjugate[t]	0	0	0
0	0	0	0	0	6 t	0	0	0	0	0	0	0	3 t Conjugate[t]	0	0
0	0	0	0	-6 t	0	0	0	0	0	0	0	0	0	3 t Conjugate[t]	0
0	0	0	0	0	0	0	-6 t	0	0	0	0	0	0	0	3 t Conjugate[t]

Pattern Hunting

We noticed in the matrix of $\Box_b^t$ on $\mathcal{H}_3(\mathbb{S}^3)$ that the entries seemed to line up in an interesting way:

```
In[43]:= MatrixForm[BuildMatrix[BoxBTU, H3]]
Out[43]//MatrixForm=
```

3	0	0	0	0	0	0	0	0	0	0	-6 Conjugate[t]	0	0	0	0
0	3	0	0	0	0	0	0	0	0	6 Conjugate[t]	0	0	0	0	0
0	0	3	0	0	0	0	0	-6 Conjugate[t]	0	0	0	0	0	0	0
0	0	0	3	0	0	0	0	0	-6 Conjugate[t]	0	0	0	0	0	0
0	0	0	0	4 + 3 t Conjugate[t]	0	0	0	0	0	0	0	0	0	-2 Conjugate[t]	0
0	0	0	0	0	4 + 3 t Conjugate[t]	0	0	0	0	0	0	0	2 Conjugate[t]	0	0
0	0	0	0	0	0	4 + 3 t Conjugate[t]	0	0	0	0	0	-2 Conjugate[t]	0	0	0
0	0	0	0	0	0	0	4 + 3 t Conjugate[t]	0	0	0	0	0	0	0	-2 Conjugate[t]
0	0	-2 t	0	0	0	0	0	3 + 4 t Conjugate[t]	0	0	0	0	0	0	0
0	0	0	-2 t	0	0	0	0	0	3 + 4 t Conjugate[t]	0	0	0	0	0	0
0	2 t	0	0	0	0	0	0	0	0	3 + 4 t Conjugate[t]	0	0	0	0	0
-2 t	0	0	0	0	0	0	0	0	0	0	3 + 4 t Conjugate[t]	0	0	0	0
0	0	0	0	0	0	-6 t	0	0	0	0	0	3 t Conjugate[t]	0	0	0
0	0	0	0	0	6 t	0	0	0	0	0	0	0	3 t Conjugate[t]	0	0
0	0	0	0	-6 t	0	0	0	0	0	0	0	0	0	3 t Conjugate[t]	0
0	0	0	0	0	0	0	-6 t	0	0	0	0	0	0	0	3 t Conjugate[t]

This means that the matrix is block diagonal, so we set out to find the block diagonal form of this matrix.

Pattern Hunting

In[50]:= `BlockDiagonalize[BoxBTUMatrix[H3]] // MatrixForm`

Out[50]//MatrixForm=

3	-2 t	0	0	0	0	0	0
-6 Conjugate[t]	3 + 4 t Conjugate[t]	0	0	0	0	0	0
0	0	3	2 t	0	0	0	0
0	0	6 Conjugate[t]	3 + 4 t Conjugate[t]	0	0	0	0
0	0	0	0	3	-2 t	0	0
0	0	0	0	-6 Conjugate[t]	3 + 4 t Conjugate[t]	0	0
0	0	0	0	0	0	3	-2 t
0	0	0	0	0	0	-6 Conjugate[t]	3 + 4 t Conjugate[t]
4 + 3 t Conjugate[t]	-6 t	0	0	0	0	0	0
-2 Conjugate[t]	3 t Conjugate[t]	0	0	0	0	0	0
0	0	4 + 3 t Conjugate[t]	6 t	0	0	0	0
0	0	2 Conjugate[t]	3 t Conjugate[t]	0	0	0	0
0	0	0	0	4 + 3 t Conjugate[t]	-6 t	0	0
0	0	0	0	-2 Conjugate[t]	3 t Conjugate[t]	0	0
0	0	0	0	0	0	4 + 3 t Conjugate[t]	-6 t
0	0	0	0	0	0	-2 Conjugate[t]	3 t Conjugate[t]

Once we figured this out, the structure of the matrix was actually surprisingly regular: our matrix consists of two pairs of almost identical blocks, with zeros everywhere else.

Pattern Hunting

```
In[50]:= BlockDiagonalize[BoxBTUMatrix[H3]] // MatrixForm
```

Out[50]//MatrixForm=

3	-2 t	0	0	0	0	0	0
-6 Conjugate[t]	3 + 4 t Conjugate[t]	0	0	0	0	0	0
0	0	3	2 t	0	0	0	0
0	0	6 Conjugate[t]	3 + 4 t Conjugate[t]	0	0	0	0
0	0	0	0	3	-2 t	0	0
0	0	0	0	-6 Conjugate[t]	3 + 4 t Conjugate[t]	0	0
0	0	0	0	0	0	3	-2 t
0	0	0	0	0	0	-6 Conjugate[t]	3 + 4 t Conjugate[t]
4 + 3 t Conjugate[t]	-6 t	0	0	0	0	0	0
-2 Conjugate[t]	3 t Conjugate[t]	0	0	0	0	0	0
0	0	4 + 3 t Conjugate[t]	6 t	0	0	0	0
0	0	2 Conjugate[t]	3 t Conjugate[t]	0	0	0	0
0	0	0	0	4 + 3 t Conjugate[t]	-6 t	0	0
0	0	0	0	-2 Conjugate[t]	3 t Conjugate[t]	0	0
0	0	0	0	0	0	4 + 3 t Conjugate[t]	-6 t
0	0	0	0	0	0	-2 Conjugate[t]	3 t Conjugate[t]

Once we figured this out, the structure of the matrix was actually surprisingly regular: our matrix consists of two pairs of almost identical blocks, with zeros everywhere else.
If we have a block diagonal matrix, this suggests that the original space splits into multiple invariant subspaces, which make up the blocks here. So what are these invariant subspaces?

Pattern Hunting

If we look back at our expansion of $\Box_b^t$, we had

$$\Box_b^t = -\frac{1+|t|^2}{(1-|t|^2)^2}(\mathcal{L}\overline{\mathcal{L}} + |t|^2\overline{\mathcal{L}}\mathcal{L} + t\mathcal{L}^2 + \overline{t}\overline{\mathcal{L}}^2)$$

We actually know that $\mathcal{L}\overline{\mathcal{L}}$ and $\overline{\mathcal{L}}\mathcal{L}$ have diagonal representations on $\mathcal{H}_k(\mathbb{S}^3)$, so what is happening with $\mathcal{L}^2$ and $\overline{\mathcal{L}}^2$?

Pattern Hunting

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We actually know that $\mathcal{L}\overline{\mathcal{L}}$ and $\overline{\mathcal{L}}\mathcal{L}$ have diagonal representations on $\mathcal{H}_k(\mathbb{S}^3)$, so what is happening with $\mathcal{L}^2$ and $\overline{\mathcal{L}}^2$? Surprisingly, $\mathcal{L}^2$ and $\overline{\mathcal{L}}^2$ take basis elements of $\mathcal{H}_k(\mathbb{S}^3)$ to other basis elements of $\mathcal{H}_k(\mathbb{S}^3)$! So our idea was to track which basis elements $\mathcal{L}^2$ and $\overline{\mathcal{L}}^2$ send to which other basis elements.

Pattern Hunting

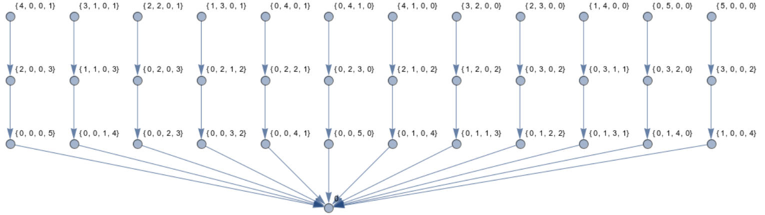
If we label a basis element by the 4-tuple $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ originally used in generating the basis for $\mathcal{H}_{p,q}(\mathbb{S}^3)$, then their action looks like this on $\mathcal{H}_5(\mathbb{S}^3)$:

Pattern Hunting

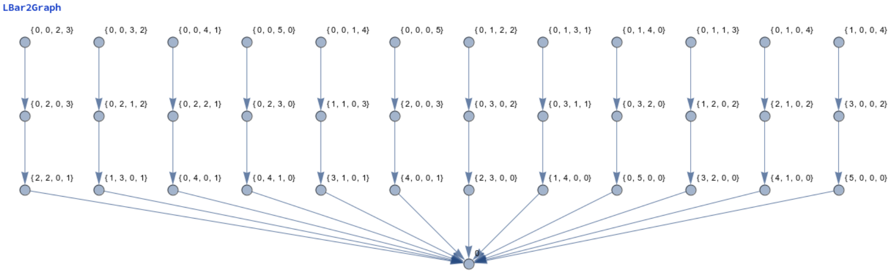
If we label a basis element by the 4-tuple $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ originally used in generating the basis for $\mathcal{H}_{p,q}(\mathbb{S}^3)$, then their action looks like this on $\mathcal{H}_5(\mathbb{S}^3)$:

```
k = 5;
L2Graph = Graph[HMappings[L2, SpecialHBasis[k]], VertexLabels → "Name"];
LBar2Graph = Graph[HMappings[LBar2, SpecialHBasis[k]], VertexLabels → "Name"];
HighlightGraph[GraphUnion[L2Graph, LBar2Graph], {L2Graph, LBar2Graph}, VertexLabels → "Name", EdgeShapeFunction → GraphElementData["HalfFilledArrow", "ArrowSize" → 0.02]]
```

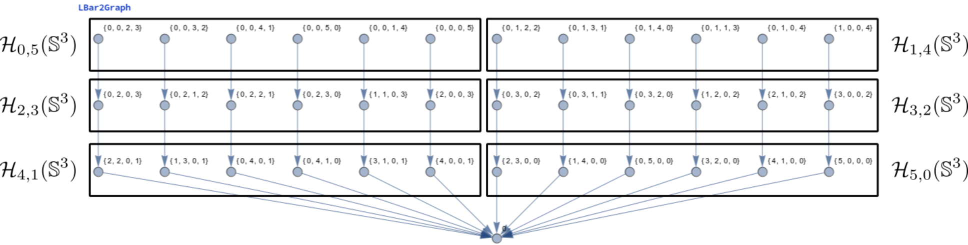
L2Graph



Pattern Hunting



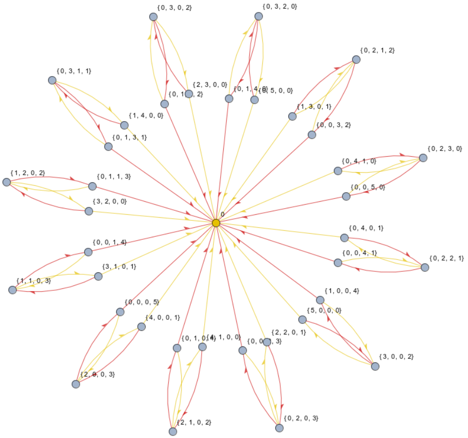
Pattern Hunting



Pattern Hunting

```
HighlightGraph[GraphUnion[L2Graph, LBar2Graph], {L2Graph, LBar2Graph}, VertexLabels → "Name", EdgeShapeFunction → GraphElementData["HalfFilledArrow", "ArrowSize" → 0.02]]
```

Out[94]=



Pattern Hunting

So we get these chains of basis elements generated by either $\mathcal{L}^2$ or $\overline{\mathcal{L}}^2$. So what do these chains look like, and what happens when we apply $\Box_b^t$ to them?

Pattern Hunting

So we get these chains of basis elements generated by either $\mathcal{L}^2$ or $\overline{\mathcal{L}}^2$. So what do these chains look like, and what happens when we apply $\Box_b^t$ to them?

```
In[38]:= BasisChains[5]
Out[38]= {{-120 Conjugate[z2]^5, -2400 Conjugate[z2]^3 z1^2, -14400 Conjugate[z2] z1^4},
  {-120 Conjugate[z1] Conjugate[z2]^4, -1440 Conjugate[z1] Conjugate[z2]^2 z1^2 + 960 Conjugate[z2]^3 z1 z2, -2880 Conjugate[z1] z1^4 + 11520 Conjugate[z2] z1^3 z2},
  {-120 Conjugate[z1]^2 Conjugate[z2]^3, -720 Conjugate[z1]^2 Conjugate[z2] z1^2 + 1440 Conjugate[z1] Conjugate[z2]^2 z1 z2 - 240 Conjugate[z2]^3 z2^2, 5760 Conjugate[z1] z1^3 z2 - 8640 Conjugate[z2] z1^2 z2^2},
  {-120 Conjugate[z1]^3 Conjugate[z2]^2, -240 Conjugate[z1]^3 z1^2 + 1440 Conjugate[z1]^2 Conjugate[z2] z1 z2 - 720 Conjugate[z1] Conjugate[z2]^2 z2^2, -8640 Conjugate[z1] z1^2 z2^2 + 5760 Conjugate[z2] z1 z2^3},
  {-120 Conjugate[z1]^4 Conjugate[z2], 960 Conjugate[z1]^3 z1 z2 - 1440 Conjugate[z1]^2 Conjugate[z2] z2^2, 11520 Conjugate[z1] z1 z2^3 - 2880 Conjugate[z2] z2^4},
  {-120 Conjugate[z1]^5, -2400 Conjugate[z1]^3 z2^2, -14400 Conjugate[z1] z2^4}, {96 Conjugate[z1] Conjugate[z2]^3 z1 - 24 Conjugate[z2]^4 z2, 576 Conjugate[z1] Conjugate[z2] z1^3 - 864 Conjugate[z2]^2 z1^2 z2, -2880 z1^4 z2},
  {72 Conjugate[z1]^2 Conjugate[z2]^2 z1 - 48 Conjugate[z1] Conjugate[z2]^3 z2, 144 Conjugate[z1]^2 z1^3 - 864 Conjugate[z1] Conjugate[z2] z1^2 z2 + 432 Conjugate[z2]^2 z1 z2^2, 2880 z1^3 z2^2},
  {48 Conjugate[z1]^3 Conjugate[z2] z1 - 72 Conjugate[z1]^2 Conjugate[z2]^2 z2, -432 Conjugate[z1]^2 z1^2 z2 + 864 Conjugate[z1] Conjugate[z2] z1 z2^2 - 144 Conjugate[z2]^2 z2^3, -2880 z1^2 z2^3},
  {24 Conjugate[z1]^4 z1 - 96 Conjugate[z1]^3 Conjugate[z2] z2, 864 Conjugate[z1]^2 z1 z2^2 - 576 Conjugate[z1] Conjugate[z2] z2^3, 2880 z1 z2^4},
  {-120 Conjugate[z1]^4 z2, -1440 Conjugate[z1]^2 z2^3, -2880 z2^5}, {-120 Conjugate[z2]^4 z1, -1440 Conjugate[z2]^2 z1^3, -2880 z1^5}}
```

Pattern Hunting

```
In[39]:= MatrixForm @* BoxBTUMatrix /@ %38
```

Out[39]=

$$\left\{\begin{pmatrix} 5 & -40\,t & 0 \\ -\text{Conjugate}[t] & 9+8\,t\,\text{Conjugate}[t] & -72\,t \\ 0 & -\text{Conjugate}[t] & 5+8\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 5 & -40\,t & 0 \\ -\text{Conjugate}[t] & 9+8\,t\,\text{Conjugate}[t] & -72\,t \\ 0 & -\text{Conjugate}[t] & 5+8\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 5 & -40\,t & 0 \\ -\text{Conjugate}[t] & 9+8\,t\,\text{Conjugate}[t] & -72\,t \\ 0 & -\text{Conjugate}[t] & 5+8\,t\,\text{Conjugate}[t] \end{pmatrix},\right.$$

$$\begin{pmatrix} 5 & -40\,t & 0 \\ -\text{Conjugate}[t] & 9+8\,t\,\text{Conjugate}[t] & -72\,t \\ 0 & -\text{Conjugate}[t] & 5+8\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 5 & -40\,t & 0 \\ -\text{Conjugate}[t] & 9+8\,t\,\text{Conjugate}[t] & -72\,t \\ 0 & -\text{Conjugate}[t] & 5+8\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 5 & -40\,t & 0 \\ -\text{Conjugate}[t] & 9+8\,t\,\text{Conjugate}[t] & -72\,t \\ 0 & -\text{Conjugate}[t] & 5+8\,t\,\text{Conjugate}[t] \end{pmatrix},$$

$$\begin{pmatrix} 8+5\,t\,\text{Conjugate}[t] & -72\,t & 0 \\ -\text{Conjugate}[t] & 8+9\,t\,\text{Conjugate}[t] & -40\,t \\ 0 & -\text{Conjugate}[t] & 5\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 8+5\,t\,\text{Conjugate}[t] & -72\,t & 0 \\ -\text{Conjugate}[t] & 8+9\,t\,\text{Conjugate}[t] & -40\,t \\ 0 & -\text{Conjugate}[t] & 5\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 8+5\,t\,\text{Conjugate}[t] & -72\,t & 0 \\ -\text{Conjugate}[t] & 8+9\,t\,\text{Conjugate}[t] & -40\,t \\ 0 & -\text{Conjugate}[t] & 5\,t\,\text{Conjugate}[t] \end{pmatrix},$$

$$\left.\begin{pmatrix} 8+5\,t\,\text{Conjugate}[t] & -72\,t & 0 \\ -\text{Conjugate}[t] & 8+9\,t\,\text{Conjugate}[t] & -40\,t \\ 0 & -\text{Conjugate}[t] & 5\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 8+5\,t\,\text{Conjugate}[t] & -72\,t & 0 \\ -\text{Conjugate}[t] & 8+9\,t\,\text{Conjugate}[t] & -40\,t \\ 0 & -\text{Conjugate}[t] & 5\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 8+5\,t\,\text{Conjugate}[t] & -72\,t & 0 \\ -\text{Conjugate}[t] & 8+9\,t\,\text{Conjugate}[t] & -40\,t \\ 0 & -\text{Conjugate}[t] & 5\,t\,\text{Conjugate}[t] \end{pmatrix}\right\}$$

They're almost the invariant subspaces we've been looking for. If we add in some constants of the form $\frac{(2k)!}{(2k-2i)!}$ to the elements of the chain where $i \in \mathbb{N}$, we get the following:

Pattern Hunting

```
In[48]:= MatrixForm @* BoxBTUMatrix /@ BasisChains[5]
```

Out[48]=

$$\left\{\begin{pmatrix} 5 & -2\,t & 0 \\ -20\,\text{Conjugate}[t] & 9+8\,t\,\text{Conjugate}[t] & -12\,t \\ 0 & -6\,\text{Conjugate}[t] & 5+8\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 5 & -2\,t & 0 \\ -20\,\text{Conjugate}[t] & 9+8\,t\,\text{Conjugate}[t] & -12\,t \\ 0 & -6\,\text{Conjugate}[t] & 5+8\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 5 & -2\,t & 0 \\ -20\,\text{Conjugate}[t] & 9+8\,t\,\text{Conjugate}[t] & -12\,t \\ 0 & -6\,\text{Conjugate}[t] & 5+8\,t\,\text{Conjugate}[t] \end{pmatrix},\right.$$

$$\begin{pmatrix} 5 & -2\,t & 0 \\ -20\,\text{Conjugate}[t] & 9+8\,t\,\text{Conjugate}[t] & -12\,t \\ 0 & -6\,\text{Conjugate}[t] & 5+8\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 5 & -2\,t & 0 \\ -20\,\text{Conjugate}[t] & 9+8\,t\,\text{Conjugate}[t] & -12\,t \\ 0 & -6\,\text{Conjugate}[t] & 5+8\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 5 & -2\,t & 0 \\ -20\,\text{Conjugate}[t] & 9+8\,t\,\text{Conjugate}[t] & -12\,t \\ 0 & -6\,\text{Conjugate}[t] & 5+8\,t\,\text{Conjugate}[t] \end{pmatrix},$$

$$\begin{pmatrix} 8+5\,t\,\text{Conjugate}[t] & -6\,t & 0 \\ -12\,\text{Conjugate}[t] & 8+9\,t\,\text{Conjugate}[t] & -20\,t \\ 0 & -2\,\text{Conjugate}[t] & 5\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 8+5\,t\,\text{Conjugate}[t] & -6\,t & 0 \\ -12\,\text{Conjugate}[t] & 8+9\,t\,\text{Conjugate}[t] & -20\,t \\ 0 & -2\,\text{Conjugate}[t] & 5\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 8+5\,t\,\text{Conjugate}[t] & -6\,t & 0 \\ -12\,\text{Conjugate}[t] & 8+9\,t\,\text{Conjugate}[t] & -20\,t \\ 0 & -2\,\text{Conjugate}[t] & 5\,t\,\text{Conjugate}[t] \end{pmatrix},$$

$$\left.\begin{pmatrix} 8+5\,t\,\text{Conjugate}[t] & -6\,t & 0 \\ -12\,\text{Conjugate}[t] & 8+9\,t\,\text{Conjugate}[t] & -20\,t \\ 0 & -2\,\text{Conjugate}[t] & 5\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 8+5\,t\,\text{Conjugate}[t] & -6\,t & 0 \\ -12\,\text{Conjugate}[t] & 8+9\,t\,\text{Conjugate}[t] & -20\,t \\ 0 & -2\,\text{Conjugate}[t] & 5\,t\,\text{Conjugate}[t] \end{pmatrix}, \begin{pmatrix} 8+5\,t\,\text{Conjugate}[t] & -6\,t & 0 \\ -12\,\text{Conjugate}[t] & 8+9\,t\,\text{Conjugate}[t] & -20\,t \\ 0 & -2\,\text{Conjugate}[t] & 5\,t\,\text{Conjugate}[t] \end{pmatrix}\right\}$$

So these are the invariant subspaces we have been looking for! So these chains are invariant subspaces of $\Box_b^t$, which make up the blocks that we saw in the original matrix. We also get that these blocks are completely identical unlike what we had before, so this seems to be the right way to look at $\mathcal{H}_k(\mathbb{S}^3)$ under $\Box_b^t$.

Pattern Hunting

This discovery led to a major theorem, but first a definition.

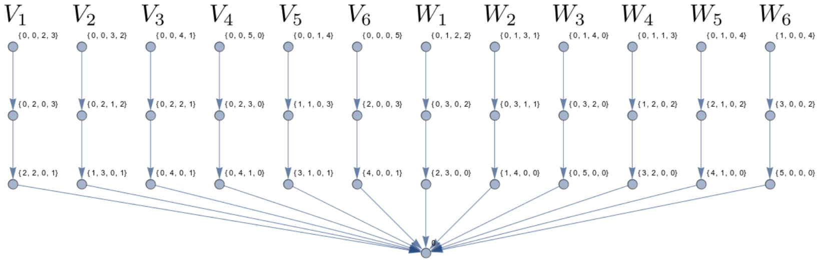
Definition 4

Let f_i be one of the $2k$ basis elements of $\mathcal{H}_{0,2k-1}(\mathbb{S}^3) \subseteq \mathcal{H}_{2k-1}(\mathbb{S}^3)$. Then we define the subspaces V_i and W_i as

$$V_i = \text{span}\{f_i, \overline{\mathcal{L}}^2 f_i, \ldots, \overline{\mathcal{L}}^{2j-2} f_i, \ldots, \overline{\mathcal{L}}^{2k-2} f_i\},$$
$$W_i = \text{span}\{\overline{\mathcal{L}} f_i, \overline{\mathcal{L}}^3 f_i, \ldots, \overline{\mathcal{L}}^{2j-1} f_i, \ldots, \overline{\mathcal{L}}^{2k-1} f_i\}.$$

This is just a more formal way of stating the first basis chains we saw earlier. We used this definition in the end because the constants used earlier complicate the definition and theorem.

Pattern Hunting



Pattern Hunting

Then the theorem is as follows:

Theorem 5

The matrix representation of $\Box_b^t$, $m(\Box_b^t)$, *on* V_i *and* W_i *is tridiagonal, where* $m(\Box_b^t)$ *on* V_i *is*

$$m(\Box_b^t) = h \begin{pmatrix} d_1 & u_1 & & & \\ -\bar{t} & d_2 & u_2 & & \\ & -\bar{t} & d_3 & \ddots & \\ & & \ddots & \ddots & u_{k-1} \\ & & & -\bar{t} & d_k \end{pmatrix}$$

where $u_j = -t \cdot (2j)(2j-1)(2k-2j)(2k-1-2j)$ *and*
$d_j = (2j-1)(2k+1-2j) + |t|^2 \cdot (2j-2)(2k+2-2j)$.

Pattern Hunting

Theorem 5

For W_i, we get something similar:

$$m(\Box_b^t) = h \begin{pmatrix} d_1 & u_1 & & & \\ -\bar{t} & d_2 & u_2 & & \\ & -\bar{t} & d_3 & \ddots & \\ & & \ddots & \ddots & u_{k-1} \\ & & & -\bar{t} & d_k \end{pmatrix}$$

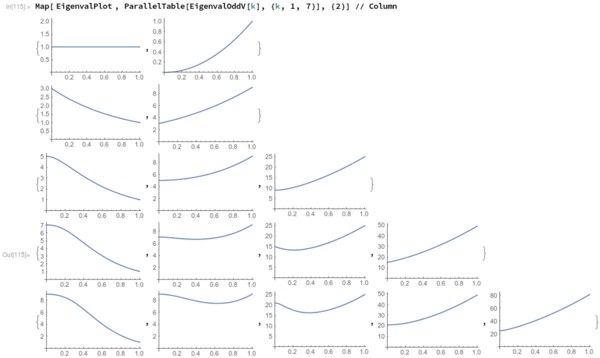
where $u_j = -t \cdot (2j+1)(2j)(2k-2j)(2k-1-2j)$ and $d_j = (2j)(2k-2j) + |t|^2 \cdot (2j-1)(2k+1-2j)$.

Finding Eigenvalues

Now that we have this concise representation of $\Box_b^t$ on $\mathcal{H}_{2k-1}(\mathbb{S}^3)$, we can start to talk about the eigenvalues of this operator. Since the eigenvalues change depending on the value of t you pick, we decided to look at the graphs of the eigenvalues as $|t|$ varies between 0 and 1.

Finding Eigenvalues

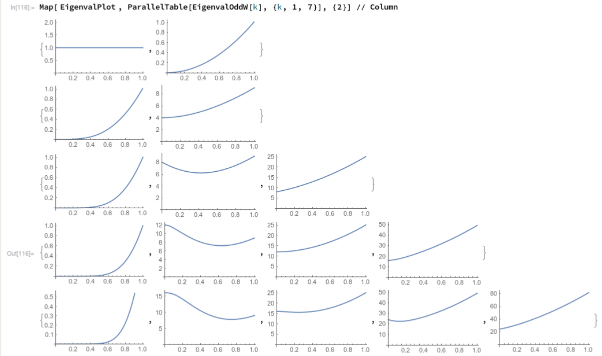
For the V_i subspaces, the graphs look like



which seem to be bounded away from 1.

Finding Eigenvalues

For the W_i subspaces, the graphs look like



Here, there is one eigenvalue that gets closer and closer to 0 as k increases. This is what we are looking for!

Finding Eigenvalues

To prove these eigenvalues exist, we first observed that our tridiagonal matrix is similar to a symmetric tridiagonal matrix: in particular,

$$\begin{pmatrix} d_1 & u_1 & & & \\ l_1 & d_2 & u_2 & & \\ & l_2 & d_3 & \ddots & \\ & & \ddots & \ddots & u_{k-1} \\ & & & l_{k-1} & d_k \end{pmatrix} \sim \begin{pmatrix} d_1 & \sqrt{u_1 l_1} & & & \\ \sqrt{u_1 l_1} & d_2 & \sqrt{u_2 l_2} & & \\ & \sqrt{u_2 l_2} & d_3 & \ddots & \\ & & \ddots & \ddots & \sqrt{u_{k-1} l_{k-1}} \\ & & & \sqrt{u_{k-1} l_{k-1}} & d_k \end{pmatrix}$$

whenever $u_i l_i > 0$.

Finding Eigenvalues

Our second observation was that we can use the Cauchy Interlacing Theorem to bound the lowest eigenvalue.

Theorem 6 (Cauchy Interlacing Theorem)

Suppose A is an $k \times k$ Hermitian matrix of rank k, and B is an $k-1 \times k-1$ matrix minor of A. If the eigenvalues of A are $\lambda_1 \leq \cdots \leq \lambda_k$ and the eigenvalues of B are $\nu_1 \leq \cdots \leq \nu_{k-1}$, then the eigenvalues of A and B interlace:

$$\lambda_1 \leq \nu_1 \leq \lambda_2 \leq \nu_2 \leq \cdots \leq \lambda_{k-1} \leq \nu_{k-1} \leq \lambda_k$$

Finding Eigenvalues

We can use this to formulate a bound on the smallest eigenvalue of the matrix: since the determinant of a matrix is the product of its eigenvalues, we have that

Lemma 7

$\lambda_1 \leq \frac{\det(A)}{\det(B)}$, where B is the upper left matrix minor of A.

Finding Eigenvalues

Since both A and B will be tridiagonal matrices, we can exploit a property of the determinant of such matrices called the continuant. The continuant is a recursive sequence: $f_1 = d_1$, and $f_i = d_{i-1}f_{i-1} - u_{i-2}l_{i-2}f_{i-2}$, where $f_0 = 1$. Then $\det(A) = f_n$ and $\det(B) = f_{n-1}$, which are the last two terms of the continuant.

Finding Eigenvalues

Applying this to our problem, we want to bound the smallest eigenvalue on the W_i spaces. If we apply all of this to $\mathcal{H}_5(\mathbb{S}^3)$, we get that

$$A = \begin{pmatrix} 8+5|t|^2 & 6\sqrt{2}|t| & 0 \\ 6\sqrt{2}|t| & 8+9|t|^2 & 2\sqrt{10}|t| \\ 0 & 2\sqrt{10}|t| & 5|t|^2 \end{pmatrix}$$

is the symmetric tridiagonal matrix on the W_i spaces, and the bound is

$$\frac{\det(A)}{\det(B)} = \frac{225|t|^6}{64+40|t|^2+45|t|^4}$$

Finding Eigenvalues

The determinant of a general tridiagonal matrix does not have a nice form, so the fact that we can express this quotient so nicely is a bit bizarre.

Finding Eigenvalues

The determinant of a general tridiagonal matrix does not have a nice form, so the fact that we can express this quotient so nicely is a bit bizarre. We found that actually,

$$\begin{aligned} 225|t|^6 &= 5|t|^2 \cdot 9|t|^2 \cdot 5|t|^2 \\ 64 &= 8 \cdot 8 \\ 40|t|^2 &= 5|t|^2 \cdot 8 \\ 45|t|^4 &= 5|t|^2 \cdot 9|t|^2 \end{aligned}$$

which means that the determinant of this matrix only depends on the main diagonal of it, which is even more bizarre.

Finding Eigenvalues

In general, the matrix representation of $\Box_b^t$ on the W_i subspaces in $\mathcal{H}_{2k-1}(\mathbb{S}^3)$ is similar to

$$A = \begin{pmatrix} a_1 + b_1|t|^2 & c_1|t| & & & \\ c_1|t| & a_2 + b_2|t|^2 & c_2|t| & & \\ & c_2|t| & a_3 + b_3|t|^2 & \ddots & \\ & & \ddots & \ddots & c_{k-1}|t| \\ & & & c_{k-1}|t| & a_k + b_k|t|^2 \end{pmatrix}$$

where $a_i = (2i)(2k-2i)$, $b_i = (2i-1)(2k+1-2i)$, and $c_i = \sqrt{(2i+1)(2i)(2k-2i)(2k-1-2i)}$.

Finding Eigenvalues

We noticed that the product of the off-diagonal entries

$$6\sqrt{2}|t| \cdot 6\sqrt{2}|t| = 8 \cdot 9|t|^2$$

or in other words,

$$c_i^2 = a_i b_{i+1}$$

Finding Eigenvalues

We noticed that the product of the off-diagonal entries

$$6\sqrt{2}|t| \cdot 6\sqrt{2}|t| = 8 \cdot 9|t|^2$$

or in other words,

$$c_i^2 = a_i b_{i+1}$$

With this observation, we noticed that this fact allows the off-diagonal entries to cancel out with some of the on-diagonal entries in the determinant, and led to the following theorem:

Theorem 8

$\det(A) = b_1 b_2 \cdots b_k |t|^{2k}$
$\det(B) = a_1 \cdots a_{k-1} + b_1 a_2 \cdots a_{k-1}|t|^2 + \cdots + b_1 \cdots b_{k-1}|t|^{2k-2}$

Finding Eigenvalues

Now, we add back our constant $\frac{1+|t|^2}{(1-|t|^2)^2}$. With some more manipulation of this final bound, we were able to arrive at our final result:

Final Result

The smallest eigenvalue of $\Box_b^t$ on $\mathcal{H}_{2k-1}(\mathbb{S}^3)$, $\lambda_{min,2k-1}$, is bounded above by

$$\lambda_{min,2k-1} \leq \frac{1+|t|^2}{(1-|t|^2)^2} \cdot \frac{\det(A)}{\det(B)} \leq \frac{1+|t|^2}{(1-|t|^2)^2}(2k-1)\sqrt{k}|t|^{2k}$$

which goes to 0 as $k \to \infty$.

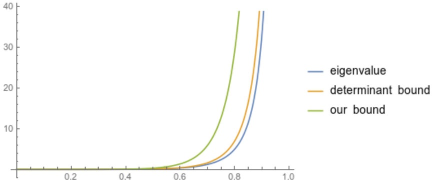
Finding Eigenvalues

In[128]:=
```
TtoXReplace[normalizer] * {EigenvalOddW[3][[1]], bounds[[3]], 5 Sqrt[3] x^6}
```

Out[128]= $\left\{\frac{(1+\text{Norm}[x]^2)\ \text{Root}\left[-225\,x^6+(64+80\,x^2+115\,x^4)\,\#1+(-16-19\,x^2)\,\#1^2+\#1^3\,\&,\,1\right]}{(1-\text{Norm}[x]^2)^2},\ \frac{225\,x^6\,(1+\text{Norm}[x]^2)}{(64+40\,x^2+45\,x^4)\,(1-\text{Norm}[x]^2)^2},\ \frac{5\sqrt{3}\,x^6\,(1+\text{Norm}[x]^2)}{(1-\text{Norm}[x]^2)^2}\right\}$

In[129]:=
```
Plot[%128, {x, 0, 1}, PlotLegends → {"eigenvalue", "determinant bound", "our bound"}]
```

Out[129]=



This is a picture of the smallest eigenvalue and both bounds on $\mathcal{H}_5(\mathbb{S}^3)$.

Acknowledgements

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