The Groupoid Interpretation of Type Theory

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What is a Groupoid?

What is a groupoid? While it can be understood as an extension of the idea of a group, it is best understood under the lens of category theory.

Definition (Category)

A category consists of two things:

- A collection of objects.
- A collection of morphisms between two objects in the category. These satisfy the following constraints:
  - If $f : A \to B$ and $g : B \to C$, there is a compsition operation yielding a morphism $g \circ f : A \to C$.
  - For every object X in the category, there is a morphism $\mathrm{id}_X : X \to X$ such that if $f : A \to X$ and $g : X \to B$ we have that $\mathrm{id}_X \circ f = f$ and $g \circ \mathrm{id}_X = g$.
  - Composition is associative: if $f : A \to B$, $g : B \to C$, and $h : C \to D$, then

$$(f \circ g) \circ h = f \circ (g \circ h).$$

What is a Groupoid?

Examples of categories include:

- Set, the category where the objects are sets and the morphisms are just normal functions between sets.
- Top, the category where the objects are topological spaces and the morphisms are just continuous functions.

Definition (Groupoid)

A groupoid is a category where for every morphism $f : A \to B$, there is an inverse morphism $f^{-1} : B \to A$ such that

$$f \circ f^{-1} = \mathrm{id}_A$$
$$f^{-1} \circ f = \mathrm{id}_B \,.$$

We also say that f is an isomorphism if it satisfies the above property.

Types are Groupoids

We can interpret the types in type theory as groupoids, where the objects are just the elements of the type and the morphisms are statements of propositional equality. This can be summed up in the following table:

Identity Type	Groupoid
$p : x =_A y$	$f : x \to y$
reflexivity	identity morphisms
symmetry	inverse morphisms
transitivity	composition of morphisms

The Fundamental Groupoid

Definition (Fundamental Groupoid)

Suppose X is a topological space. Then we define the fundamental groupoid $\pi_1(X)$ as the set of all paths between any two points of the space, where paths are considered equivalent if there is a homotopy between them, and where we can concatenate paths if the end point of the first path is the start point of the second.

The Fundamental Groupoid

This satisfies all the properties of a groupoid, and homotopy type theory comes from the idea of considering types to be fundamental groupoids of topological spaces. We have the following correspondence:

Groupoid	Fundamental Groupoid
$f : x \to y$	path from x to y
identity morphisms	constant path
inverse morphisms	reversed path
composition of morphisms	concatenation of paths

∞-groupoids

Definition (∞-groupoid)

An ∞-groupoid has the following:

- A collection of objects.
- Morphisms between objects, or 1-morphisms.
- Morphisms between 1-morphisms, or 2-morphisms.
- ...
- Morphisms between k-morphisms, or $(k+1)$-morphisms.

Each collection of k-morphisms satisfy the usual properties of morphisms in a groupoid.

∞-groupoids

We can think of higher order statements like identity types on identity types as higher order morphisms/paths in groupoids, like the following:

Identity Type	∞-Groupoid	Fundamental ∞-Groupoid
$p =_{(x =_A y)} q$	2-morphisms	homotopies
$r =_{(p =_{(x =_A y)} q)} s$	3-morphisms	homotopies of homotopies

Properties of the Identity Type

Recall that path induction is the following statement: suppose we have

$$C : \prod_{x,y:A} (x =_A y) \to \mathcal{U}$$

$$c : \prod_{x:A} C(x, x, \mathsf{refl}_x).$$

Then there is a function J such that

$$J(C, c) : \prod_{(x,y:A)} \prod_{(p:x=_A y)} C(x, y, p)$$

$$\text{where } J(C, c, x, x, \mathsf{refl}_x) = c(x)$$

The simplified version of this is "If you want to prove something is true when x and y are propositionally equal, it suffices to prove it for the case where y is x and the proof of equality is reflexivity".

Properties of the Identity Type

One can prove the following things via path induction:

- Symmetry and transitivity hold for propositional equality. Their correspondences as paths (inverses and composition) obey the usual laws.
- If $f : A \to B$ and $p : x =_A y$, then there is an induced path $f(p) : f(x) =_B f(y)$.

Properties of the Identity Type

Lemma (Transport)

Suppose $p : x =_A y$ and $P : A \to \mathcal{U}$. Then we can construct

$$p_\star : P(x) \to P(y).$$

The reason we need this lemma is to prove the statement about function types with dependent function types. A priori, if $f : \prod_{(x:A)} P(x)$ and $p : x =_A y$, then there isn't a way to define a path between $f(x)$ and $f(y)$ since $f(x) : P(x)$ and $f(y) : P(y)$ are different types. But with the Transport Lemma, we can use $p_\star$ to coerce $f(x)$ to be of type $P(y)$, so we get a path

$$f'(p) : p_\star(f(x)) =_{B(y)} f(y).$$

Uniqueness of Identity Proofs

Statement (Uniqueness of Identity Proofs)

Suppose $p, q : x =_A y$. Then we have that the type

$$p =_{(x =_A y)} q$$

is inhabited.

This statement is also abbreviated UIP, and has a couple equivalent formulations.

Statement (Equivalent Formulation)

Suppose that $s : x =_A x$. Then we have that the type

$$s =_{(x =_A x)} \mathsf{refl}_x$$

is inhabited.

Uniqueness of Identity Proofs

We also have one more equivalent formulation called Axiom K, as follows:

Statement (Axiom K)

Suppose that

$$C : \prod_{x:A} (x =_A x) \to \mathcal{U}$$

$$d : \prod_{x:A} C(x, \mathsf{refl}_x).$$

Then there is a function K such that

$$K(C, d) : \prod_{(x:A)} \prod_{(p:x=_A x)} C(x, p)$$

$$\textit{where } K(C, d, x, p) = d(p).$$

Uniqueness of Identity Proofs

As it turns out, UIP is not actually an implication of intensional type theory, and it is the case that there are models of type theory where UIP doesn't hold.

This was proven using the groupoid interpretation, where we construct types as groupoids, and UIP is the statement that between every pair of objects in the groupoid there is at most 1 morphism between them. So if we can construct types as arbitrary groupoids, then the falsity of UIP would follow from the fact that not every groupoid is like this.

Uniqueness of Identity Proofs

The identity type in the groupoid interpretation is constructed where if A is a groupoid, $x =_A y$ is the discrete groupoid on the collection of morphisms between x and y, $\Delta(A(x, y))$. Then if it is the case that $A(x, y)$ has more than 1 element, we have that there are two different elements of $\Delta(A(x, y))$ that are not equivalent to each other since the discrete groupoid only contains the identity morphisms, proving that UIP is uninhabited.

Uniqueness of Identity Proofs

As it turns out, UIP is undesirable for two main reasons:

- UIP is incompatible with the Univalence Principle.
- UIP turns our intensional type theory into an extensional type theory.